My First Difficult Olympiad Geometry Problem

Problem statement

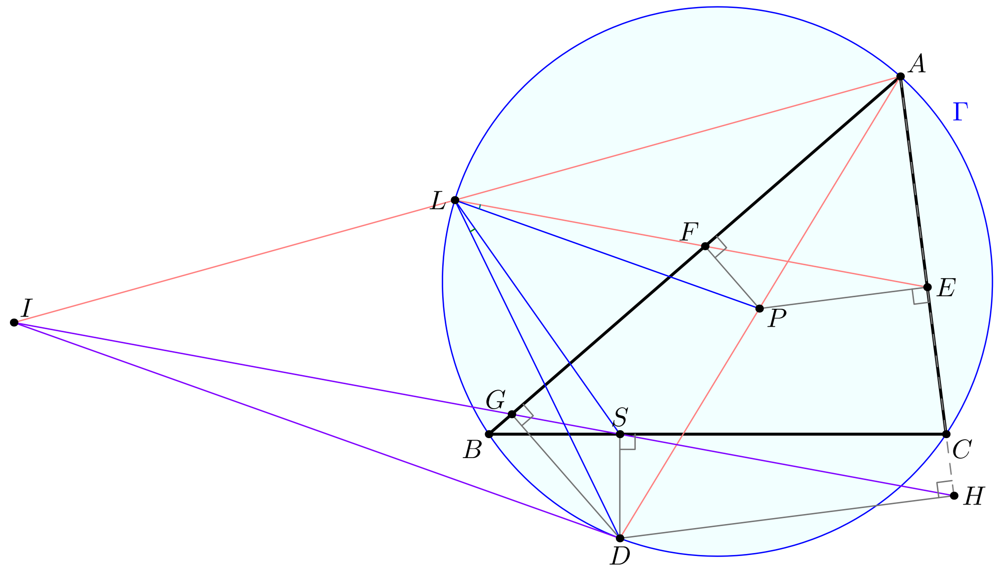
Source: We Love Geometry WeChat public account, Teacher Yao Jiabin.

Let $\triangle ABC$ be inscribed in a circle Γ. Let P be an interior point of $\triangle ABC$.

Let E and F be the feet of the perpendiculars from P to AC and AB, respectively. Let L be the intersection of line EF with Γ shown in the figure. Let line AP meet Γ again at D. Finally, let S be the foot of the perpendicular from D to BC.

Prove that

$$\angle PLE = \angle SLD.$$



Proof. Let G and H be the feet of the perpendiculars from D to AB and AC, respectively. Thus

$$DG \perp AB, \qquad DH \perp AC.$$

Since D lies on the circumcircle of $\triangle ABC$, by the Simson line theorem, the three feet G, S, H are collinear. All ratios of collinear segments below are understood as directed ratios.

Let

$$I := AL \cap GH.$$

We first claim that $LP \parallel ID$. Indeed, since

$$PF \perp AB, \quad DG \perp AB,$$

we have $PF \parallel DG$. Hence

$$\triangle AFP \sim \triangle AGD,$$

and therefore

$$\frac{AF}{AG} = \frac{AP}{AD}.$$

Similarly, since

$$PE \perp AC, \quad DH \perp AC,$$

we have $PE \parallel DH$. Hence
$$\triangle AEP \sim \triangle AHD,$$
and thus
$$\frac{AE}{AH} = \frac{AP}{AD}.$$
Consequently,
$$\frac{AF}{AG} = \frac{AE}{AH},$$
which implies
$$EF \parallel GH.$$

Since L, E, F are collinear and I, G, H are collinear, this gives
$$LE \parallel IH.$$

Moreover, applying the intercept theorem in triangle AIG, with $L \in AI$, $F \in AG$, and $LF \parallel IG$, gives
$$\frac{AL}{AI} = \frac{AF}{AG}.$$
Therefore
$$\frac{AL}{AI} = \frac{AP}{AD}.$$
Since A, L, I are collinear and A, P, D are collinear, it follows that
$$LP \parallel ID.$$

Now
$$LE \parallel IH, \qquad LP \parallel ID, \qquad EP \parallel HD.$$
Hence
$$\triangle ELP \sim \triangle HID.$$
In particular,
$$\measuredangle PLE = \measuredangle DIH.$$

It remains to prove that
$$\measuredangle DIH = \measuredangle DLS.$$
Since A, L, D, C are concyclic and A, L, I are collinear, we have
$$\measuredangle ILD = \measuredangle ALD = \measuredangle ACD.$$

Also,
$$DS \perp BC, \qquad DH \perp AC,$$
so
$$\angle DSC = \angle DHC = 90^\circ.$$
Thus S, C, H, D are concyclic. Hence
$$\measuredangle HSD = \measuredangle HCD.$$

Since A, C, H are collinear, we have
$$\measuredangle ACD = \measuredangle HCD.$$

Since I, S, H are collinear, we have

$$\measuredangle ISD = \measuredangle HSD.$$

Therefore

$$\measuredangle ACD = \measuredangle ISD.$$

Together with $\measuredangle ILD = \measuredangle ACD$, this gives

$$\measuredangle ILD = \measuredangle ISD.$$

Thus I, L, D, S are concyclic.

It follows that

$$\measuredangle DLS = \measuredangle DIS.$$

Since I, S, H are collinear, we have

$$\measuredangle DIS = \measuredangle DIH.$$

Combining this with $\measuredangle PLE = \measuredangle DIH$, we obtain

$$\measuredangle PLE = \measuredangle DLS.$$

Therefore, in ordinary angles,

$$\angle PLE = \angle SLD.$$

This proves the desired result. $\square$