Notes for Math 214, Spring 2022

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Based on Richard Bamler's Math 214 lectures and uploaded handwritten notes from Spring 2022. The mathematical content, exposition, notation, and structure are due to Richard Bamler. I copied the handwritten notes for personal study and used AI tools to assist with conversion into LaTeX. The LaTeX preamble and document style are adapted from one of Evan Chen's IMO solution TeX files. These notes are unofficial, and any errors introduced during copying, AI conversion, or checking are my own.

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§1.1 Real Projective Space

Definition 1.1. The real projective space $\mathbb{RP}^n$ is defined by

$$\mathbb{RP}^n := \left(\mathbb{R}^{n+1} \setminus \{0\}\right) / \sim,$$

where

$$x \sim y \iff x = \lambda y \text{ for some } \lambda \in \mathbb{R} \setminus \{0\}.$$

Equivalently,

$$\mathbb{RP}^n = \left\{L \subset \mathbb{R}^{n+1} \mid L \text{ is a line through the origin}\right\}.$$

Let

$$\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$$

be the quotient map. For $x \in \mathbb{R}^{n+1} \setminus \{0\}$, we write

$$\pi(x) = [x].$$

The quotient topology on $\mathbb{RP}^n$ is defined as follows:

$$A \subset \mathbb{RP}^n \text{ is open } \iff \pi^{-1}(A) \text{ is open in } \mathbb{R}^{n+1} \setminus \{0\}.$$

Equivalently, one can also describe $\mathbb{RP}^n$ as

$$S^n/(x \sim -x).$$

The quotient map is

$$\pi : S^n \to \mathbb{RP}^n, \qquad x \mapsto [x].$$

This gives the same quotient topology on $\mathbb{RP}^n$.

Definition 1.2. A point of $\mathbb{RP}^n$ is written in homogeneous coordinates as

$$[x^1 : x^2 : \cdots : x^{n+1}].$$

By definition,

$$[x^1 : x^2 : \cdots : x^{n+1}] = [\lambda x^1 : \lambda x^2 : \cdots : \lambda x^{n+1}]$$

for every $\lambda \in \mathbb{R} \setminus \{0\}$.

For each $i = 1, \ldots, n+1$, define

$$U_i := \left\{[x^1 : \cdots : x^{n+1}] \in \mathbb{RP}^n \mid x^i \neq 0\right\}.$$

The set U_i is open in $\mathbb{RP}^n$. Indeed,

$$\pi^{-1}(U_i) = \left\{x \in \mathbb{R}^{n+1} \setminus \{0\} \mid x^i \neq 0\right\},$$

which is open in $\mathbb{R}^{n+1} \setminus \{0\}$. Hence U_i is open by the definition of the quotient topology.

Moreover, the sets $U_1, \ldots, U_{n+1}$ cover $\mathbb{RP}^n$. Indeed, for every

$$[x^1 : \cdots : x^{n+1}] \in \mathbb{RP}^n,$$

not all coordinates $x^1, \ldots, x^{n+1}$ are zero. Hence at least one coordinate x^i is nonzero, so the point lies in U_i.

Definition 1.3. For each $i = 1, \dots, n+1$, define

$$\varphi_i : U_i \to \mathbb{R}^n$$

by

$$\varphi_i\left([x^1 : \cdots : x^{n+1}]\right) = \left(\frac{x^1}{x^i}, \dots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \dots, \frac{x^{n+1}}{x^i}\right).$$

The map φ_i is well-defined. Indeed, if

$$[x^1 : \cdots : x^{n+1}] = [\lambda x^1 : \cdots : \lambda x^{n+1}], \qquad \lambda \neq 0,$$

then for every $j \neq i$,

$$\frac{\lambda x^j}{\lambda x^i} = \frac{x^j}{x^i}.$$

The inverse map is

$$\varphi_i^{-1}(u^1, \dots, u^n) = [u^1 : \cdots : u^{i-1} : 1 : u^i : \cdots : u^n].$$

Therefore φ_i is a bijection. Moreover, both φ_i and φ_i^{-1} are continuous, so φ_i is a homeomorphism.

Hence

$$(U_i, \varphi_i)$$

is a coordinate chart on $\mathbb{RP}^n$.

Theorem 1.4

The collection

$$\mathcal{A} = \{(U_i, \varphi_i) \mid i = 1, \dots, n+1\}$$

is a smooth atlas on $\mathbb{RP}^n$.

Proof. The sets $U_1, \dots, U_{n+1}$ cover $\mathbb{RP}^n$, so it remains to check that the charts are smoothly compatible.

Let $i \neq j$. On the overlap $U_i \cap U_j$, both x^i and x^j are nonzero. Consider the transition map

$$\varphi_j \circ \varphi_i^{-1}.$$

In the i-th chart, a point of U_i has the form

$$[x^1 : \cdots : x^{n+1}] = \left[\frac{x^1}{x^i} : \cdots : \frac{x^{i-1}}{x^i} : 1 : \frac{x^{i+1}}{x^i} : \cdots : \frac{x^{n+1}}{x^i}\right].$$

Passing to the j-th chart means dividing every coordinate by the j-th coordinate. Hence the transition map is given by rational functions whose denominators are nonzero on the overlap.

Therefore

$$\varphi_j \circ \varphi_i^{-1}$$

is smooth. Similarly,

$$\varphi_i \circ \varphi_j^{-1}$$

is smooth. Thus the charts are smoothly compatible.

Therefore $\mathcal{A}$ is a smooth atlas on $\mathbb{RP}^n$. $\square$

§1.2 Polar Coordinates and Transition Maps

Example 1.5

Let $M = \mathbb{R}^2$.

Consider polar coordinates on

$$U = \mathbb{R}^2 \setminus (\mathbb{R}_{\geq 0} \times \{0\})\,.$$

For $z \in U$, write $z = (z^1, z^2)$. Define

$$\varphi : U \to \mathbb{R}^2$$

by

$$\varphi(z) = \big(r(z), \theta(z)\big) = \big(\varphi^1(z), \varphi^2(z)\big) = \big(|z|, \arg z\big),$$

where

$$r, \theta : U \to \mathbb{R}$$

are the coordinate functions.

Here the image of φ is

$$\varphi(U) = \mathbb{R}_+ \times (0, 2\pi).$$

Thus the polar coordinate chart is

$$(U, \varphi).$$

Example 1.6

Now consider the Euclidean coordinates on

$$V = \mathbb{R}^2.$$

Define

$$\psi : V \to \mathbb{R}^2$$

by

$$\psi(z) = \big(x(z), y(z)\big) = \big(\psi^1(z), \psi^2(z)\big) = (z^1, z^2),$$

where

$$x, y : \mathbb{R}^2 \to \mathbb{R}$$

are the coordinate functions.

Thus the Euclidean coordinate chart is

$$(V, \psi).$$

The transition map from polar coordinates to Euclidean coordinates is

$$\psi \circ \varphi^{-1} : \mathbb{R}_+ \times (0, 2\pi) \to \mathbb{R}^2 \setminus (\mathbb{R}_{\geq 0} \times \{0\})\,.$$

Explicitly,

$$(\psi \circ \varphi^{-1})(\widehat{r}, \widehat{\theta}) = \big(\widehat{r} \cos \widehat{\theta}, \widehat{r} \sin \widehat{\theta}\big).$$

Since the component functions

$$(\widehat{r}, \widehat{\theta}) \mapsto \widehat{r} \cos \widehat{\theta} \quad \text{and} \quad (\widehat{r}, \widehat{\theta}) \mapsto \widehat{r} \sin \widehat{\theta}$$

are smooth, the transition map

$$\psi \circ \varphi^{-1}$$

is smooth.

§1.3 Smooth Compatibility

Definition 1.7. Let (U, φ) and (V, ψ) be two charts on a topological manifold M. We say that (U, φ) and (V, ψ) are **smoothly compatible** if both transition maps

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

and

$$\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$$

are smooth maps between open subsets of Euclidean spaces.

> **Remark 1.8.** At this stage, we have not yet defined what it means for a map between manifolds to be smooth. Therefore we do not say that a chart itself is smooth. Instead, we say that two charts are smoothly compatible.

> **Remark 1.9.** If two charts (U, φ) and (V, ψ) are smoothly compatible, then the transition maps
>
> $$\psi \circ \varphi^{-1} \quad \text{and} \quad \varphi \circ \psi^{-1}$$
>
> are diffeomorphisms between open subsets of Euclidean spaces.

§1.4 Smooth Atlases and Maximal Atlases

Definition 1.10. An **atlas** $\mathcal{A}$ of an n-dimensional topological manifold M is a collection of charts

$$\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$$

such that

$$M = \bigcup_{\alpha \in I} U_\alpha.$$

Definition 1.11. An atlas $\mathcal{A}$ is called a **smooth atlas** if all charts in $\mathcal{A}$ are pairwise smoothly compatible.

Definition 1.12. A smooth atlas $\mathcal{A}$ is called a **maximal smooth atlas** if there is no smooth atlas $\mathcal{A}'$ such that

$$\mathcal{A} \subsetneq \mathcal{A}'.$$

Equivalently, $\mathcal{A}$ is maximal if every chart smoothly compatible with all charts in $\mathcal{A}$ is already contained in $\mathcal{A}$.

Theorem 1.13

Every smooth atlas $\mathcal{A}$ of M is contained in a unique maximal smooth atlas.

Proof. We first prove existence.

Define

$$\overline{\mathcal{A}} := \left\{ (U,\varphi) \;\middle|\; \begin{array}{l} (U,\varphi) \text{ is a chart of } M, \\ (U,\varphi) \text{ is smoothly compatible with every chart in } \mathcal{A} \end{array} \right\}.$$

Clearly,

$$\mathcal{A} \subset \overline{\mathcal{A}}.$$

Claim — $\overline{\mathcal{A}}$ is a smooth atlas.

Proof. Since $\mathcal{A} \subset \overline{\mathcal{A}}$ and $\mathcal{A}$ covers M, the collection $\overline{\mathcal{A}}$ also covers M.

It remains to prove that any two charts in $\overline{\mathcal{A}}$ are smoothly compatible.

Let

$$(U_1,\varphi_1),(U_2,\varphi_2) \in \overline{\mathcal{A}}.$$

If

$$U_1 \cap U_2 = \varnothing,$$

then the transition maps have empty domain, so the two charts are smoothly compatible vacuously.

Now suppose

$$U_1 \cap U_2 \neq \varnothing.$$

For each point

$$p \in U_1 \cap U_2,$$

since $\mathcal{A}$ is an atlas, there exists a chart

$$(V,\psi) \in \mathcal{A}$$

such that

$$p \in V.$$

By definition of $\overline{\mathcal{A}}$, both (U_1,φ_1) and (U_2,φ_2) are smoothly compatible with (V,ψ).

Therefore the transition maps

$$\psi \circ \varphi_1^{-1} \quad \text{and} \quad \varphi_2 \circ \psi^{-1}$$

are smooth on their domains. On the open set

$$\varphi_1(U_1 \cap U_2 \cap V),$$

we have

$$\varphi_2 \circ \varphi_1^{-1} = \left(\varphi_2 \circ \psi^{-1}\right) \circ \left(\psi \circ \varphi_1^{-1}\right).$$

Hence $\varphi_2 \circ \varphi_1^{-1}$ is smooth near $\varphi_1(p)$.

Since this holds for every $p \in U_1 \cap U_2$, the transition map

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is smooth.

Similarly,

$$\varphi_1 \circ \varphi_2^{-1}$$

is smooth. Therefore (U_1,φ_1) and (U_2,φ_2) are smoothly compatible.

Thus $\overline{\mathcal{A}}$ is a smooth atlas. □

Now we prove maximality.

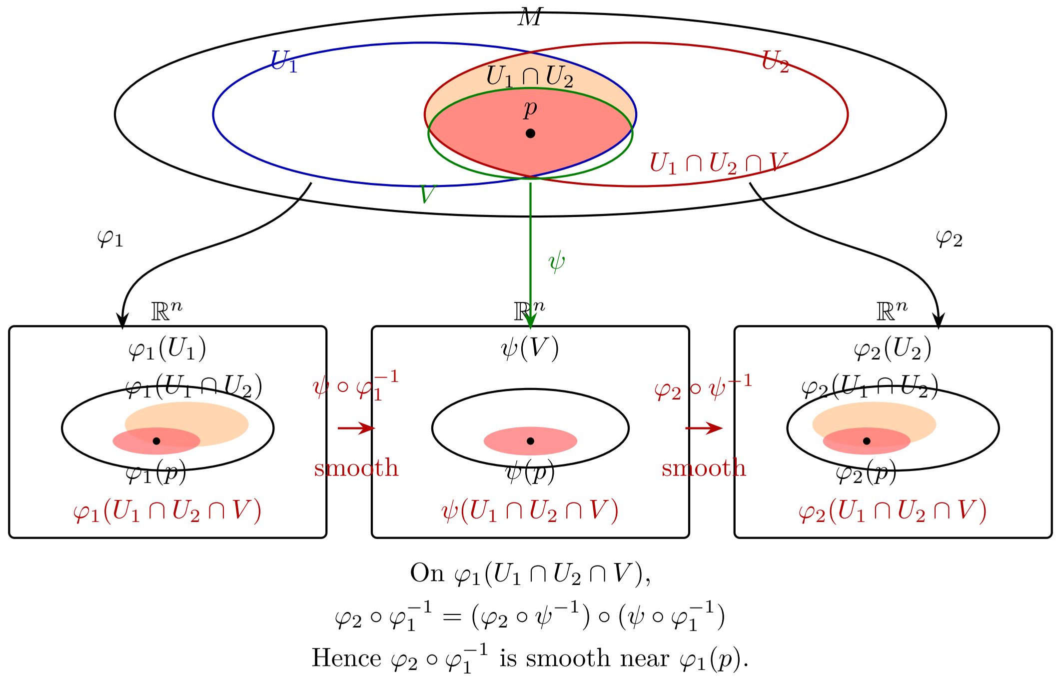


Figure 1: The local factorization used in the proof of the first claim of Theorem 1.13.

Claim — If $\mathcal{A}'$ is a smooth atlas with $\mathcal{A} \subset \mathcal{A}'$, then $\mathcal{A}' \subset \overline{\mathcal{A}}$.

Proof. Let

$$(U', \varphi') \in \mathcal{A}'.$$

Since $\mathcal{A}'$ is a smooth atlas, (U', φ') is smoothly compatible with every chart in $\mathcal{A}'$. Since

$$\mathcal{A} \subset \mathcal{A}',$$

it follows that (U', φ') is smoothly compatible with every chart in $\mathcal{A}$.

Therefore, by definition of $\overline{\mathcal{A}}$,

$$(U', \varphi') \in \overline{\mathcal{A}}.$$

Hence

$$\mathcal{A}' \subset \overline{\mathcal{A}}.$$

□

If there were a smooth atlas $\mathcal{B}$ such that

$$\overline{\mathcal{A}} \subsetneq \mathcal{B},$$

then $\mathcal{A} \subset \mathcal{B}$, so by the previous claim we would have

$$\mathcal{B} \subset \overline{\mathcal{A}},$$

a contradiction. Hence $\overline{\mathcal{A}}$ is maximal.

It remains to prove uniqueness.

Suppose $\mathcal{B}$ is another maximal smooth atlas containing $\mathcal{A}$. Since $\mathcal{A} \subset \mathcal{B}$, the previous claim gives

$$\mathcal{B} \subset \overline{\mathcal{A}}.$$

But $\mathcal{B}$ is maximal and $\overline{\mathcal{A}}$ is a smooth atlas. Hence

$$\mathcal{B} = \overline{\mathcal{A}}.$$

Therefore the maximal smooth atlas containing $\mathcal{A}$ is unique. $\square$

> **Remark 1.14.** Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are two smooth atlases on M such that every chart in $\mathcal{A}_1$ is smoothly compatible with every chart in $\mathcal{A}_2$. Then
>
> $$\overline{\mathcal{A}_1} = \overline{\mathcal{A}_2}.$$
>
> Indeed, under this assumption,
>
> $$\mathcal{A}_1 \cup \mathcal{A}_2$$
>
> is a smooth atlas. Since it contains both $\mathcal{A}_1$ and $\mathcal{A}_2$, the uniqueness of maximal smooth atlases gives
>
> $$\overline{\mathcal{A}_1} = \overline{\mathcal{A}_1 \cup \mathcal{A}_2} = \overline{\mathcal{A}_2}.$$

§2 2026-05-14

§2.1 Smooth Structures and Smooth Manifolds

Definition 2.1. Let M be a topological manifold. A **smooth structure** on M is a maximal smooth atlas $\mathcal{A}$ on M.

Definition 2.2. A **smooth manifold** is a pair

$$(M^n, \mathcal{A}),$$

where M^n is a topological n-manifold and $\mathcal{A}$ is a smooth structure on M.

Usually we use the shorthand notation

$$M = (M^n, \mathcal{A}).$$

If

$$(U, \varphi) \in \mathcal{A},$$

then (U, φ) is called a smooth chart of M.

> **Remark 2.3.** The same construction can be modified by replacing "smooth" with other regularity classes. For example, one can define
>
> $$C^k\text{-manifolds}, \qquad C^\omega\text{-manifolds},$$
>
> where C^ω means real analytic.

§2.2 Local Properties of Smooth Charts

Exercise 2.4. Let $(U, \varphi) \in \mathcal{A}$, and let $U' \subset U$ be open. Then

$$(U', \varphi|_{U'}) \in \mathcal{A}.$$

Proof. Since $U' \subset U$ is open, $\varphi(U')$ is open in $\mathbb{R}^n$, and

$$\varphi|_{U'} : U' \to \varphi(U')$$

is a homeomorphism. Thus $(U', \varphi|_{U'})$ is a chart.

It is smoothly compatible with every chart in $\mathcal{A}$, because its transition maps are restrictions of the corresponding transition maps for (U, φ). Since $\mathcal{A}$ is maximal, it follows that

$$(U', \varphi|_{U'}) \in \mathcal{A}.$$

□

Exercise 2.5. Let $(U, \varphi) \in \mathcal{A}$, and let

$$\chi : \varphi(U) \to \chi(\varphi(U)) \subset \mathbb{R}^n$$

be a diffeomorphism onto an open subset of $\mathbb{R}^n$. Then

$$(U, \chi \circ \varphi) \in \mathcal{A}.$$

Proof. The map

$$\chi \circ \varphi : U \to \chi(\varphi(U))$$

is a homeomorphism, hence $(U, \chi \circ \varphi)$ is a chart.

For any $(V, \psi) \in \mathcal{A}$, the transition map

$$\psi \circ (\chi \circ \varphi)^{-1} = \psi \circ \varphi^{-1} \circ \chi^{-1}$$

is smooth, because it is a composition of smooth maps. Similarly,

$$(\chi \circ \varphi) \circ \psi^{-1} = \chi \circ \varphi \circ \psi^{-1}$$

is smooth. Hence $(U, \chi \circ \varphi)$ is smoothly compatible with all charts in $\mathcal{A}$. By maximality,

$$(U, \chi \circ \varphi) \in \mathcal{A}.$$

□

Exercise 2.6. Let $\varphi : U \to \mathbb{R}^n$ be an injective map, where $U \subset M$ is open. Suppose that for every $p \in U$, there exists an open neighborhood $U_p \subset U$ of p such that

$$(U_p, \varphi|_{U_p}) \in \mathcal{A}.$$

Then

$$(U, \varphi) \in \mathcal{A}.$$

Proof. The condition is local. Since each $\varphi|_{U_p}$ is a coordinate map, the image $\varphi(U_p)$ is open in $\mathbb{R}^n$. Because

$$\varphi(U) = \bigcup_{p \in U} \varphi(U_p),$$

the image $\varphi(U)$ is open in $\mathbb{R}^n$.

Moreover, φ is a homeomorphism onto its image because this can be checked locally on the open cover $\{U_p\}_{p \in U}$. Thus (U, φ) is a chart.

It remains to check smooth compatibility. Let $(V, \psi) \in \mathcal{A}$. For each $p \in U \cap V$, choose U_p as above. On

$$\varphi(U_p \cap V),$$

the transition map

$$\psi \circ \varphi^{-1}$$

is equal to

$$\psi \circ (\varphi|_{U_p})^{-1},$$

which is smooth because $(U_p, \varphi|_{U_p}) \in \mathcal{A}$. Hence $\psi \circ \varphi^{-1}$ is smooth locally, and therefore smooth. The other transition map is treated similarly.

Thus (U, φ) is smoothly compatible with all charts in $\mathcal{A}$. By maximality,

$$(U, \varphi) \in \mathcal{A}.$$

□

Example 2.7

Let $M = \mathbb{R}$, and let $\mathcal{A}$ be the standard smooth structure, namely the maximal smooth atlas containing

$$(\mathbb{R}, \mathrm{id}_{\mathbb{R}}).$$

Let $\mathcal{A}^*$ be the maximal smooth atlas containing

$$(\mathbb{R}, \varphi), \qquad \varphi(x) = x^3.$$

Then

$$\mathcal{A}^* \neq \mathcal{A}.$$

Indeed, the two charts $(\mathbb{R}, \mathrm{id}_{\mathbb{R}})$ and $(\mathbb{R}, \varphi)$ are not smoothly compatible, because one transition map is

$$\mathrm{id}_{\mathbb{R}} \circ \varphi^{-1}(x) = \sqrt[3]{x},$$

which is not smooth at 0.

However, the smooth manifolds

$$(\mathbb{R}, \mathcal{A}) \qquad \text{and} \qquad (\mathbb{R}, \mathcal{A}^*)$$

are diffeomorphic. For example, the map

$$\varphi : (\mathbb{R}, \mathcal{A}^*) \to (\mathbb{R}, \mathcal{A}), \qquad x \mapsto x^3,$$

is a diffeomorphism.

Remark 2.8. This example shows that two smooth structures on the same underlying topological manifold may be different, even though the resulting smooth manifolds can still be diffeomorphic.

§2.3 Standard Examples and Open Submanifolds

Example 2.9

The standard smooth structure on $\mathbb{R}^n$ is the maximal smooth atlas containing

$$(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n}).$$

Theorem 2.10

Let $(M, \mathcal{A})$ be a smooth manifold, and let

$$M' \subset M$$

be open. Define

$$\mathcal{A}' := \{(U, \varphi) \in \mathcal{A} \mid U \subset M'\}.$$

Then

$$(M', \mathcal{A}')$$

is also a smooth manifold.

Proof. Since $M' \subset M$ is open, it is a topological manifold with the subspace topology.

The charts in $\mathcal{A}'$ cover M'. Indeed, if $p \in M'$, choose a chart $(U, \varphi) \in \mathcal{A}$ with $p \in U$. Then

$$U \cap M'$$

is open in U, so by the restriction property,

$$(U \cap M', \varphi|_{U \cap M'}) \in \mathcal{A}.$$

Moreover its domain is contained in M', hence it belongs to $\mathcal{A}'$.

Smooth compatibility is inherited from $\mathcal{A}$. Therefore $\mathcal{A}'$ is a smooth atlas on M'. Taking the maximal smooth atlas determined by $\mathcal{A}'$, we obtain the induced smooth structure on M'. $\square$

Example 2.11

Let S^n be the unit sphere. Its standard smooth structure is the maximal smooth atlas containing the stereographic projection charts

$$(U_N, \varphi_N), \qquad (U_S, \varphi_S),$$

where $U_N = S^n \setminus \{N\}$ and $U_S = S^n \setminus \{S\}$.

Equivalently, it is the maximal smooth atlas containing the usual graph charts

$$(U_i^{\pm}, \varphi_i^{\pm}).$$

Example 2.12

Let V be an n-dimensional real vector space. Define

$$\mathcal{A}' := \{(V, \varphi) \mid \varphi : V \to \mathbb{R}^n \text{ is a vector space isomorphism}\}.$$

Then $\mathcal{A}'$ is a smooth atlas, because the transition maps are linear isomorphisms

$$\mathbb{R}^n \to \mathbb{R}^n,$$

hence smooth. The standard smooth structure on V is the maximal smooth atlas containing $\mathcal{A}'$.

Remark 2.13. In general, if $\mathcal{A}$ is the maximal smooth atlas containing a smooth atlas $\mathcal{A}'$, and if a chart (U, φ) is smoothly compatible with every chart in $\mathcal{A}'$, then

$$\mathcal{A}' \cup \{(U, \varphi)\}$$

is still a smooth atlas. Hence

$$(U, \varphi) \in \mathcal{A}.$$

Remark 2.14. Reading examples include

$$S^n, \qquad \mathbb{RP}^n, \qquad \mathrm{GL}(n, \mathbb{R}),$$

and product manifolds.

Remark 2.15. Digression.

Does every topological manifold have a smooth structure?

$$\begin{aligned} n \leq 3 : &\quad \text{Yes, and unique up to diffeomorphism;} \\ n \geq 4 : &\quad \text{No, and smooth structure may not be unique;} \end{aligned}$$

Remark 2.16. Subquestion.

Are there exotic smooth structures on S^n?

Here "exotic" means not diffeomorphic to the standard one.

$$\begin{aligned} n \leq 3 : &\quad \text{no;} \\ n = 4 : &\quad \text{unknown; } \textit{Smooth Poincaré Conjecture} \\ n \geq 5 : &\quad \text{almost completely understood.} \end{aligned}$$

On spheres are related to the group of homotopy spheres Θ_n which can be characterized using algebraic topology. case $n = 126$ is often noted as unclear in this context.

§2.4 Smooth Manifold Construction Lemma

Theorem 2.17 (Smooth Manifold Construction Lemma)
Let M be a set, and let

$$\{\varphi_\alpha : U_\alpha \to \mathbb{R}^n\}_{\alpha \in I}$$

be a collection of injective maps, where $U_\alpha \subset M$. Suppose the following conditions hold.

1. For every $\alpha, \beta \in I$,
$$\varphi_\alpha(U_\alpha \cap U_\beta)$$
is open in $\mathbb{R}^n$.
2. For every $\alpha, \beta \in I$, the transition map
$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$
is smooth.
3. The set M is covered by countably many of the sets U_α.
4. For any $p, q \in M$ with $p \neq q$, either there exists some $\alpha \in I$ such that
$$p, q \in U_\alpha,$$
or there exist $\alpha, \beta \in I$ such that
$$p \in U_\alpha, \qquad q \in U_\beta, \qquad U_\alpha \cap U_\beta = \varnothing.$$

Then M has a unique topology and a unique smooth structure such that all pairs

$$(U_\alpha, \varphi_\alpha)$$

are smooth charts.

Proof. Define a topology on M by declaring that a subset $A \subset M$ is open if and only if

$$\varphi_\alpha(A \cap U_\alpha)$$

is open in $\mathbb{R}^n$ for every $\alpha \in I$.

With this topology, each U_α is open in M, and each

$$\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$$

is a homeomorphism onto an open subset of $\mathbb{R}^n$.

The first two assumptions imply that the charts $(U_\alpha, \varphi_\alpha)$ are smoothly compatible. The countability assumption gives second-countability. The separation assumption gives the Hausdorff property. Therefore M becomes a topological n-manifold.

Finally, let $\mathcal{A}$ be the maximal smooth atlas containing all charts

$$(U_\alpha, \varphi_\alpha).$$

This gives the desired smooth structure. Uniqueness follows because the topology and the maximal smooth atlas are forced by the given charts. $\square$

§2.5 Grassmannians

Example 2.18 (Grassmannians)
Let
$$1 \leq k \leq n.$$
The Grassmannian is
$$\mathrm{Gr}_k(\mathbb{R}^n) := \{V \subset \mathbb{R}^n \mid V \text{ is a linear subspace and } \dim V = k\}.$$
In particular,
$$\mathrm{Gr}_1(\mathbb{R}^n) = \mathbb{RP}^{n-1}.$$

Let
$$I := \left\{ (P, Q) \;\middle|\; \begin{array}{l} P, Q \subset \mathbb{R}^n \text{ are linear subspaces,} \\ \mathbb{R}^n = P \oplus Q,\ \dim P = k,\ \dim Q = n-k \end{array} \right\}.$$
For
$$\alpha = (P, Q) \in I,$$
define
$$U_\alpha := \{V \in \mathrm{Gr}_k(\mathbb{R}^n) \mid V \cap Q = \{0\}\}.$$

Lemma 2.19
For every $V \in U_\alpha$, there exists a unique linear map
$$A_{P,Q,V} : P \to Q$$
such that
$$V = \{x + A_{P,Q,V}x \mid x \in P\}.$$

Proof. Since
$$\mathbb{R}^n = P \oplus Q,$$
every vector in $\mathbb{R}^n$ can be written uniquely as $x + y$ with $x \in P$ and $y \in Q$.

If $V \in U_\alpha$, then
$$V \cap Q = \{0\}.$$
The projection
$$\pi_P : V \to P$$
is injective, because if $\pi_P(v) = 0$, then $v \in Q$, hence $v \in V \cap Q = \{0\}$. Since
$$\dim V = \dim P = k,$$
the map $\pi_P : V \to P$ is an isomorphism.

Therefore, for each $x \in P$, there is a unique vector $v \in V$ with $\pi_P(v) = x$. Write
$$v = x + A_{P,Q,V}x$$
with $A_{P,Q,V}x \in Q$. This defines a unique linear map
$$A_{P,Q,V} : P \to Q.$$
Hence
$$V = \{x + A_{P,Q,V}x \mid x \in P\}.$$
$\square$

Define

$$\varphi_\alpha : U_\alpha \to L(P,Q)$$

by

$$\varphi_\alpha(V) = A_{P,Q,V}.$$

Since

$$L(P,Q) \cong \mathbb{R}^{k(n-k)}$$

after choosing bases of P and Q, this gives a coordinate chart on the Grassmannian.

§2.6 Manifolds with Boundary

Definition 2.20. Let

$$\mathbb{H}^n := \{x = (x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}$$

be the closed upper half-space. Its boundary is

$$\partial\mathbb{H}^n := \{x \in \mathbb{H}^n \mid x^n = 0\}.$$

Definition 2.21. Let M be an n-manifold with boundary. A point $p \in M$ is called a **boundary point** if there exists a chart

$$(U, \varphi)$$

such that

$$p \in U \qquad \text{and} \qquad \varphi(p) \in \partial\mathbb{H}^n.$$

The set of all boundary points is denoted by

$$\partial M.$$

For manifolds with boundary, transition maps are maps between open subsets of $\mathbb{H}^n$. Their smoothness means that all partial derivatives exist and extend smoothly up to the boundary. Equivalently, locally they extend to smooth maps on open subsets of $\mathbb{R}^n$.

Lemma 2.22 (Invariance of boundary)
Let $A, B \subset \mathbb{H}^n$ be open subsets in the subspace topology, and let

$$F : A \to B$$

be a diffeomorphism. Then

$$F(A \cap \partial\mathbb{H}^n) = B \cap \partial\mathbb{H}^n.$$

Proof. It suffices to prove that

$$F(A \cap \partial\mathbb{H}^n) \subset B \cap \partial\mathbb{H}^n,$$

because the reverse inclusion follows by applying the same argument to F^{-1}.

Suppose, for contradiction, that there exists

$$x \in A \cap \partial\mathbb{H}^n$$

such that
$$y := F(x) \in \operatorname{Int} \mathbb{H}^n.$$
Let
$$G := F^{-1}.$$
Since y is an interior point of $\mathbb{H}^n$, there is an ordinary open neighborhood $W \subset \mathbb{R}^n$ of y such that
$$W \subset B \cap \operatorname{Int} \mathbb{H}^n.$$
Write
$$G = (G^1, \dots, G^n).$$
Because $G(W) \subset \mathbb{H}^n$, we have
$$G^n(z) \geq 0$$
for all $z \in W$. Moreover,
$$G^n(y) = x^n = 0,$$
because $x \in \partial\mathbb{H}^n$. Thus G^n has a local minimum at y. Hence
$$dG^n_y = 0.$$
Therefore the image of dG_y is contained in the hyperplane
$$\mathbb{R}^{n-1} \times \{0\},$$
so
$$\operatorname{rank} dG_y \leq n - 1.$$

On the other hand, since F and G are inverse diffeomorphisms, the derivative dG_y must be invertible. This is a contradiction. Hence $F(x)$ cannot be an interior point of $\mathbb{H}^n$, and therefore
$$F(x) \in \partial\mathbb{H}^n.$$
□

Lemma 2.23

If $p \in M$ is a boundary point, then for every chart (U, φ) containing p,
$$\varphi(p) \in \partial\mathbb{H}^n.$$

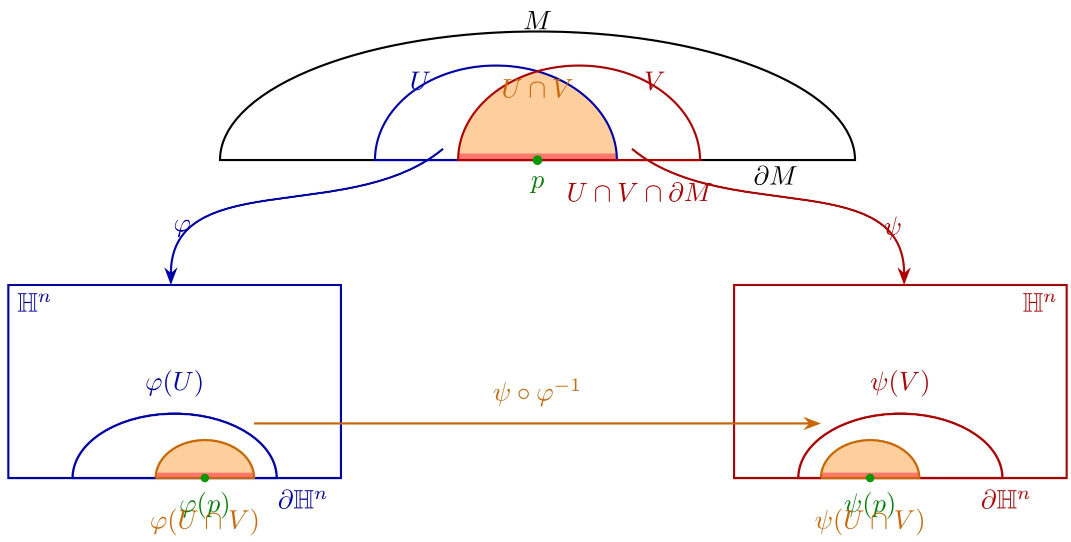
Proof. Since p is a boundary point, by definition there exists a chart (V, ψ) containing p such that
$$\psi(p) \in \partial\mathbb{H}^n.$$
Let (U, φ) be any chart containing p. On the overlap $U \cap V$, the transition map
$$\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$$
is a diffeomorphism between open subsets of $\mathbb{H}^n$. By the invariance of boundary for diffeomorphisms between open subsets of $\mathbb{H}^n$, it maps boundary points of $\mathbb{H}^n$ to boundary points of $\mathbb{H}^n$.

Since
$$\psi(p) \in \partial\mathbb{H}^n,$$
we obtain
$$\varphi(p) = (\varphi \circ \psi^{-1})(\psi(p)) \in \partial\mathbb{H}^n.$$
Therefore every chart containing p sends p to $\partial\mathbb{H}^n$. □

§3 2026-05-16

§3.1 Boundary Points and Manifold Boundary

Recap: manifolds with boundary, transition map



Theorem 3.1 (Boundary points are well-defined)
Let M be a smooth manifold with boundary, and let $p \in M$. Suppose

$$(U, \varphi), \qquad (V, \psi)$$

are two smooth charts containing p. Then

$$\varphi(p) \in \partial\mathbb{H}^n \iff \psi(p) \in \partial\mathbb{H}^n.$$

Equivalently,

$$\varphi(p) \in \operatorname{Int}\mathbb{H}^n \iff \psi(p) \in \operatorname{Int}\mathbb{H}^n.$$

Proof. Consider the transition map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V).$$

Since (U, φ) and (V, ψ) are smoothly compatible charts, this is a smooth transition map between open subsets of $\mathbb{H}^n$.

Moreover, it restricts to a smooth map

$$\psi \circ \varphi^{-1} : \partial\mathbb{H}^n \cap \varphi(U \cap V) \to \partial\mathbb{H}^n \cap \psi(U \cap V).$$

Thus the transition map sends boundary points of $\mathbb{H}^n$ to boundary points of $\mathbb{H}^n$.

Therefore,

$$\varphi(p) \in \partial\mathbb{H}^n \iff \psi(p) \in \partial\mathbb{H}^n.$$

Similarly,

$$\varphi(p) \in \operatorname{Int}\mathbb{H}^n \iff \psi(p) \in \operatorname{Int}\mathbb{H}^n.$$

Hence whether p is a boundary point is independent of the chart. □

Definition 3.2. Let M be a smooth manifold with boundary. The **interior** of M is

$$\operatorname{Int} M := \{p \in M \mid p \text{ is an interior point}\}.$$

The **boundary** of M is

$$\partial M := \{p \in M \mid p \text{ is a boundary point}\}.$$

Thus

$$M = \operatorname{Int} M \sqcup \partial M.$$

> **Remark 3.3.** The interior of M is a smooth n-manifold without boundary:
>
> $$\operatorname{Int} M \quad \text{is a smooth } n\text{-manifold without boundary.}$$
>
> The boundary of M is a smooth $(n-1)$-manifold without boundary:
>
> $$\partial M \quad \text{is a smooth } (n-1)\text{-manifold without boundary.}$$

> **Remark 3.4.** The topological interior and the manifold interior need not be the same notion. Similarly, the topological boundary and the manifold boundary need not agree.

Example 3.5

Let

$$M = \{x \in \mathbb{R}^n \mid x^n > 0\} \subset \mathbb{R}^n.$$

Then M has no manifold boundary:

$$\partial M = \varnothing.$$

However, as a subset of $\mathbb{R}^n$, it has topological boundary

$$\partial_{\text{top}} M = \mathbb{R}^{n-1} \times \{0\}.$$

Example 3.6

If M is an n-manifold with boundary, then

$$\partial_{\text{top}} M = \varnothing$$

when M is regarded as a topological space in itself. Indeed,

$$\operatorname{Int}_{\text{top}} M = M.$$

But the manifold boundary ∂M may be nonempty.

Example 3.7

Let

$$M = S^n \subset \mathbb{R}^{n+1}.$$

Then

$$\partial_{\mathrm{mf}} M = \varnothing,$$

and

$$\partial_{\mathrm{top}} M = S^n$$

if S^n is viewed as a subset of $\mathbb{R}^{n+1}$.

Example 3.8

Let

$$M = \mathbb{H}^n \cap B(0,1).$$

Then M is a smooth manifold with boundary. Its manifold boundary is

$$\partial M = \{x \in B(0,1) \mid x^n = 0\}.$$

Geometrically, this is the flat part of the boundary.

§3.2 Smoothness on Half-Spaces

Remark 3.9. The notion of smoothness for maps defined on subsets of $\mathbb{H}^n$ is the following. All partial derivatives should exist and be continuous up to the boundary.

Definition 3.10. Let

$$A \subset \mathbb{H}^n$$

be an arbitrary subset, not necessarily open. A map

$$f : A \to \mathbb{R}^m$$

is called **smooth** if it can be extended to a smooth map

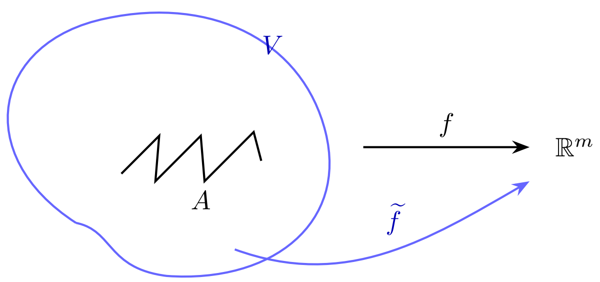
$$\widetilde{f} : V \to \mathbb{R}^m$$

where

$$A \subset V \subset \mathbb{R}^n$$

and V is open, such that

$$\widetilde{f}|_A = f.$$



Theorem 3.11 (Seeley's Extension Theorem)
Let
$$U \subset \mathbb{H}^n$$
be open in the relative topology, and let
$$f : U \to \mathbb{R}^m$$
be a map. Suppose that, for every multi-index I, the partial derivative
$$\partial^I f$$
exists on $U \cap \operatorname{Int} \mathbb{H}^n$ and extends continuously to all of U. Then there exist an open set
$$\widetilde{U} \subset \mathbb{R}^n, \qquad U \subset \widetilde{U},$$
and a smooth map
$$\widetilde{f} : \widetilde{U} \to \mathbb{R}^m$$
such that
$$\widetilde{f}|_U = f.$$

In other words, a map on an open subset of $\mathbb{H}^n$ is smooth up to the boundary exactly when it can be extended smoothly across the boundary hyperplane $\partial\mathbb{H}^n$.

Definition 3.12 (Partial derivatives and multi-index notation)**.** Let
$$W \subset \mathbb{R}^n$$
be open, and let
$$f : W \to \mathbb{R}^m.$$
For each coordinate direction e_i, the **partial derivative** of f in the x^i-direction is defined by
$$\frac{\partial f}{\partial x^i}(x) := \lim_{h \to 0} \frac{f(x + he_i) - f(x)}{h},$$
provided the limit exists.

A **multi-index** is an n-tuple
$$I = (i_1, \dots, i_n), \qquad i_1, \dots, i_n \in \mathbb{Z}_{\geq 0}.$$
Its order is
$$|I| = i_1 + \cdots + i_n.$$
We write
$$\partial^I f := \frac{\partial^{|I|} f}{\partial (x^1)^{i_1} \cdots \partial (x^n)^{i_n}}.$$
For example, if $n = 3$ and $I = (2, 0, 1)$, then
$$\partial^I f = \frac{\partial^3 f}{\partial (x^1)^2 \partial x^3}.$$

Remark 3.13. The condition that $\partial^I f$ extends continuously to U means the following. First compute $\partial^I f$ on the interior part

$$U \cap \operatorname{Int} \mathbb{H}^n.$$

Then, for every boundary point $p \in U \cap \partial\mathbb{H}^n$, the values of $\partial^I f(x)$ must have a limit as x approaches p from inside $\mathbb{H}^n$. This limiting value is declared to be the value of the continuous extension at p.

§3.3 Smooth Maps Between Manifolds

Definition 3.14. Let M be a smooth manifold, possibly with boundary, and let N be a smooth manifold. A map

$$f : M \to N$$

is called **smooth** if for every point $p \in M$, there exist smooth charts

$$(U, \varphi) \quad \text{of } M, \qquad (V, \psi) \quad \text{of } N,$$

such that

$$p \in U, \qquad f(U) \subset V,$$

and the coordinate representation

$$\widehat{f} := \psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V) \subset \mathbb{R}^m$$

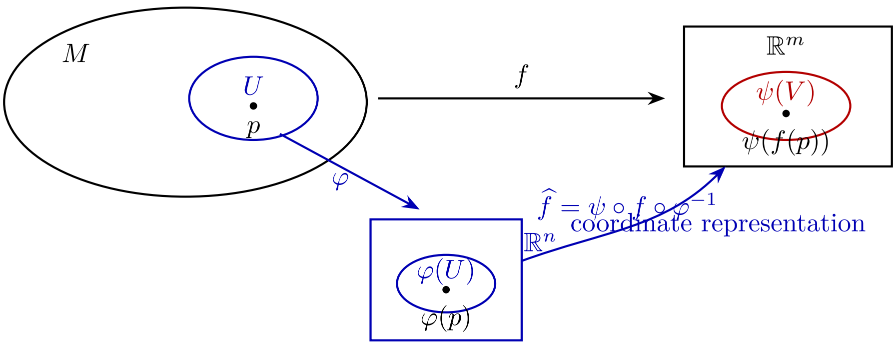
is smooth.

The map

$$\widehat{f} = \psi \circ f \circ \varphi^{-1}$$

is called the **coordinate representation** of f with respect to the charts

$$(U, \varphi) \quad \text{and} \quad (V, \psi).$$



Thus

$$C^\infty(M, N) := \{f : M \to N \mid f \text{ is smooth}\}.$$

Lemma 3.15

If
$$f : M \to N$$
is smooth, then for any smooth charts
$$(U, \varphi) \quad \text{of } M, \qquad (V, \psi) \quad \text{of } N,$$
the coordinate representation
$$\psi \circ f \circ \varphi^{-1} : \varphi\big(U \cap f^{-1}(V)\big) \to \psi(V)$$
is smooth.

Proof. Let
$$p \in U \cap f^{-1}(V).$$
Since f is smooth, there exist smooth charts
$$(\widetilde{U}, \widetilde{\varphi}) \quad \text{of } M, \qquad (\widetilde{V}, \widetilde{\psi}) \quad \text{of } N,$$
such that
$$p \in \widetilde{U}, \qquad F(\widetilde{U}) \subset \widetilde{V},$$
and
$$\widetilde{\psi} \circ F \circ \widetilde{\varphi}^{-1} : \widetilde{\varphi}(\widetilde{U}) \to \widetilde{\psi}(\widetilde{V})$$
is smooth.

On the domain where all expressions are defined, we have
$$\psi \circ F \circ \varphi^{-1} = \left(\psi \circ \widetilde{\psi}^{-1}\right) \circ \left(\widetilde{\psi} \circ F \circ \widetilde{\varphi}^{-1}\right) \circ \left(\widetilde{\varphi} \circ \varphi^{-1}\right).$$
Here
$$\widetilde{\varphi} \circ \varphi^{-1}$$
and
$$\psi \circ \widetilde{\psi}^{-1}$$
are smooth transition maps, while
$$\widetilde{\psi} \circ F \circ \widetilde{\varphi}^{-1}$$
is smooth by assumption. Therefore
$$\psi \circ F \circ \varphi^{-1}$$
is smooth near $\varphi(p)$.

Since p was arbitrary, the coordinate representation
$$\psi \circ F \circ \varphi^{-1} : \varphi\big(U \cap F^{-1}(V)\big) \to \psi(V)$$
is smooth. $\square$

§3.4 Basic properties and definitions

Example 3.16

Let

$$f : M \to \mathbb{R}$$

be a smooth function.

A smooth function can describe, for example, a "temperature distribution" on M, or a "smooth curve" if

$$f : [0, 1] \to M.$$

Remark 3.17. Read the basic properties and definitions of smooth maps.

§3.5 Partitions of Unity

Example 3.18

Given

$$f_-, f_+ \in C^\infty(\mathbb{R}),$$

we want to find

$$f \in C^\infty(\mathbb{R})$$

such that

$$f = f_- \quad \text{on } (-\infty, -1),$$

and

$$f = f_+ \quad \text{on } (1, \infty).$$

Choose a cutoff function

$$\psi_- \in C^\infty(\mathbb{R})$$

such that

$$\psi_- = 1 \quad \text{on } (-\infty, -1],$$

and

$$\operatorname{supp} \psi_- \subset (-\infty, 1).$$

Set

$$\psi_+ := 1 - \psi_-.$$

Then define

$$f := \psi_- f_- + \psi_+ f_+.$$

Informally,

$$\psi_- + \psi_+ \equiv 1,$$

with

$$\operatorname{supp} \psi_- \subset (-\infty, 1), \qquad \operatorname{supp} \psi_+ \subset (-1, \infty).$$

This is a simple instance of a partition of unity subordinate to the cover

$$\{(-\infty, 1), (-1, \infty)\}.$$

Definition 3.19. Let

$$\mathcal{X} = \{X_\alpha\}_{\alpha \in A}$$

be an open cover of a topological space X. A **partition of unity subordinate to $\mathcal{X}$** is a family of continuous functions

$$\{\psi_\alpha : X \to \mathbb{R}\}_{\alpha \in A}$$

such that:

1. $$0 \le \psi_\alpha \le 1 \qquad \text{for all } \alpha \in A.$$

2. $$\operatorname{supp} \psi_\alpha \subset X_\alpha.$$

3. The family
$$\{\operatorname{supp} \psi_\alpha\}_{\alpha \in A}$$
is locally finite.

4. $$\sum_{\alpha \in A} \psi_\alpha(x) \equiv 1 \qquad \text{for all } x \in X.$$

Remark 3.20. The local finiteness condition implies that

$$\sum_{\alpha \in A} \psi_\alpha$$

is locally a finite sum. Since

$$\sum_{\alpha \in A} \psi_\alpha \equiv 1,$$

we also have

$$X = \bigcup_{\alpha \in A} \operatorname{supp} \psi_\alpha.$$

Theorem 3.21 (Existence of smooth partitions of unity)
Let M be a smooth manifold, possibly with boundary. For any open cover

$$\mathcal{X} = \{X_\alpha\}_{\alpha \in A}$$

of M, there exists a smooth partition of unity

$$\{\psi_\alpha\}_{\alpha \in A}$$

subordinate to $\mathcal{X}$.

Remark 3.22. Equivalently, for any open cover of a smooth manifold, one can find smooth bump functions whose supports lie inside the elements of the cover and whose locally finite sum is identically equal to 1.

§4 2026-05-19

§4.1 Existence of Partitions of Unity

Theorem 4.1

Let M be a smooth manifold, possibly with boundary. Let

$$\mathcal{X} = \{X_\alpha\}_{\alpha \in A}$$

be an indexed open cover of M. Then there exists a smooth partition of unity subordinate to $\mathcal{X}$.

Proof. For simplicity, assume first that M has no boundary.

Since each X_α is open in M, it is a smooth manifold in its own right. Hence each X_α has a basis of regular coordinate balls. Therefore there exists a basis

$$\mathcal{B}_\alpha$$

of regular coordinate balls for X_α.

Thus

$$\bigcup_{\alpha \in A} \mathcal{B}_\alpha$$

is a basis for M. Since M is a smooth manifold, it is paracompact. Hence the open cover $\mathcal{X}$ has a countable locally finite refinement

$$\{B_i\}_{i=1}^{\infty}$$

consisting of elements of

$$\bigcup_{\alpha \in A} \mathcal{B}_\alpha.$$

We may choose these balls so that the family of closures

$$\{\overline{B_i}\}_{i=1}^{\infty}$$

is also locally finite.

For each B_i, choose an index

$$\alpha(i) \in A$$

such that

$$B_i \subset X_{\alpha(i)}.$$

Choose a smaller coordinate ball

$$B_i' \subset B_i$$

such that

$$\overline{B_i'} \subset B_i.$$

Since B_i is a coordinate ball, choose a smooth coordinate map

$$\varphi_i : B_i \to \mathbb{R}^n$$

such that

$$\varphi_i(B_i) = B_{r_i}(0), \qquad \varphi_i(B_i') = B_{r_i'}(0), \qquad r_i' < r_i.$$

Define a smooth function

$$f_i : M \to \mathbb{R}$$

by

$$f_i(p) = \begin{cases} H_i(\varphi_i(p)), & p \in B_i, \\ 0, & p \in M \setminus B_i, \end{cases}$$

where

$$H_i : \mathbb{R}^n \to \mathbb{R}$$

is a smooth bump function satisfying

$$H_i > 0 \quad \text{on } \varphi_i(B_i'),$$

and

$$H_i = 0 \quad \text{on } \mathbb{R}^n \setminus \varphi_i(B_i).$$

Then f_i is well-defined and smooth, and

$$\operatorname{supp} f_i \subset \overline{B_i} \subset X_{\alpha(i)}.$$

Now define

$$f : M \to \mathbb{R}$$

by

$$f(p) = \sum_i f_i(p).$$

Since the family

$$\{\overline{B_i}\}_{i=1}^{\infty}$$

is locally finite, the sum has only finitely many nonzero terms in a neighborhood of each point. Hence f is smooth.

Moreover, the sets B_i' still cover M, so for every $p \in M$ there exists some i such that

$$p \in B_i',$$

and therefore

$$f_i(p) > 0.$$

Thus

$$f(p) > 0$$

for every $p \in M$.

Define

$$g_i(p) := \frac{f_i(p)}{f(p)}.$$

Then each g_i is well-defined and smooth, and

$$0 \leq g_i(p) \leq 1.$$

Moreover,

$$\sum_i g_i(p) = \frac{\sum_i f_i(p)}{f(p)} = 1.$$

Also,

$$\operatorname{supp} g_i = \operatorname{supp} f_i \subset \overline{B_i} \subset X_{\alpha(i)}.$$

For each $\alpha \in A$, define

$$\chi_\alpha : M \to \mathbb{R}$$

by

$$\chi_\alpha := \sum_{\alpha(i)=\alpha} g_i.$$

Since the family $\{\operatorname{supp} g_i\}$ is locally finite, this sum is locally finite. Hence each χ_α is smooth.

Furthermore,

$$\operatorname{supp} \chi_\alpha \subset \bigcup_{\alpha(i)=\alpha} \operatorname{supp} g_i \subset X_\alpha,$$

and

$$\sum_{\alpha \in A} \chi_\alpha = \sum_i g_i = 1.$$

Thus

$$\{\chi_\alpha\}_{\alpha \in A}$$

is the desired smooth partition of unity subordinate to $\mathcal{X}$. □

§4.2 Complex Projective Space

Example 4.2 (Complex projective spaces)

Let

$$\mathbb{C}^{n+1}_* := \mathbb{C}^{n+1} \setminus \{0\}.$$

Define an equivalence relation on $\mathbb{C}^{n+1}_*$ by

$$z \sim \lambda z, \qquad \lambda \in \mathbb{C}^*.$$

The quotient space is called the **complex projective space**

$$\mathbb{C}P^n := \mathbb{C}^{n+1}_* / \sim .$$

Equivalently,

$$\mathbb{C}P^n = \{[z] \mid z \in \mathbb{C}^{n+1}_*\}.$$

For

$$z = (z^1, \dots, z^{n+1}) \in \mathbb{C}^{n+1}_*,$$

we write

$$[z] = [z^1 : z^2 : \cdots : z^{n+1}].$$

Thus

$$[z^1 : \cdots : z^{n+1}] = [\lambda z^1 : \cdots : \lambda z^{n+1}]$$

for all

$$\lambda \in \mathbb{C}^*.$$

For each

$$i = 1, \dots, n+1,$$

define

$$U_i := \{[z] \in \mathbb{C}P^n \mid z^i \neq 0\}.$$

Then

$$\{U_i\}_{i=1}^{n+1}$$

is an open cover of $\mathbb{C}P^n$.

Define

$$\varphi_i : U_i \to \mathbb{C}^n$$

by

$$\varphi_i([z]) = \left(\frac{z^1}{z^i}, \ldots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \ldots, \frac{z^{n+1}}{z^i}\right).$$

This is well-defined because multiplying z by a nonzero complex scalar does not change any quotient

$$\frac{z^j}{z^i}.$$

The inverse map is given by

$$\varphi_i^{-1}(w^1, \ldots, w^n) = [w^1 : \cdots : w^{i-1} : 1 : w^i : \cdots : w^n].$$

Hence each

$$(U_i, \varphi_i)$$

is a chart for $\mathbb{C}P^n$.

Remark 4.3. The space $\mathbb{C}P^n$ is naturally a complex manifold of complex dimension n. Equivalently, it is a smooth real manifold of real dimension

$$2n.$$

Claim — The charts

$$(U_i, \varphi_i), \qquad i = 1, \ldots, n+1,$$

make $\mathbb{C}P^n$ into a complex manifold.

Proof. We need to check that the transition maps are holomorphic.

For

$$i, j \in \{1, \ldots, n+1\},$$

we have

$$U_i \cap U_j = \{[z] \in \mathbb{C}P^n \mid z^i \neq 0,\ z^j \neq 0\}.$$

The transition map is

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j).$$

Assume, for simplicity, that

$$i < j.$$

Let

$$(w^1, \ldots, w^n) \in \varphi_i(U_i \cap U_j).$$

Under the inverse of the i-th chart, this point corresponds to

$$[z] = [w^1 : \cdots : w^{i-1} : 1 : w^i : \cdots : w^n].$$

In homogeneous coordinates, this means

$$z^i = 1,$$

and the other coordinates are the entries of

$$(w^1, \ldots, w^n).$$

Since the point lies in

$$U_i \cap U_j,$$

we also have

$$z^j \neq 0.$$

Passing from the i-th chart to the j-th chart means rescaling the homogeneous coordinates by dividing every coordinate by z^j. Thus

$$[z^1 : \cdots : z^{i-1} : 1 : z^{i+1} : \cdots : z^j : \cdots : z^{n+1}]$$

is rewritten as

$$\left[\frac{z^1}{z^j} : \cdots : \frac{z^{i-1}}{z^j} : \frac{1}{z^j} : \frac{z^{i+1}}{z^j} : \cdots : 1 : \cdots : \frac{z^{n+1}}{z^j}\right].$$

Therefore the transition map

$$\varphi_j \circ \varphi_i^{-1}$$

is given componentwise by functions of the form

$$\frac{z^k}{z^j} \quad \text{or} \quad \frac{1}{z^j}.$$

Equivalently, in the i-th affine coordinates, the components are functions of the form

$$\frac{w^a}{w^b} \quad \text{or} \quad \frac{1}{w^b},$$

on the open set where

$$w^b \neq 0.$$

These functions are holomorphic. Hence

$$\varphi_j \circ \varphi_i^{-1}$$

is holomorphic.

Similarly,

$$\varphi_i \circ \varphi_j^{-1}$$

is holomorphic. Therefore the charts are holomorphically compatible.

Hence $\mathbb{C}P^n$ is a complex manifold. □

> **Remark 4.4.** Since every holomorphic map is smooth as a real map, the same atlas also gives $\mathbb{C}P^n$ the structure of a smooth real manifold of real dimension
>
> $$2n.$$

Definition 4.5. The smooth structure on $\mathbb{C}P^n$ determined by the atlas

$$\{(U_i, \varphi_i)\}_{i=1}^{n+1}$$

is called the standard smooth structure on complex projective space.